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Yang–Mills equations and the inverse scattering transform

Javier Villarroel†

Program in Applied Mathematics, University of Colorado at Boulder, Boulder, CO 80309, USA

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Abstract. We consider the self-dual Yang-Mills system along with its hyperbolic version. We solve the corresponding initial value problem in the second case, and a boundary value problem for self-dual Yang-Mills equation.

1. Introduction

As is well known, the self-dual Yang-Mills equations in Euclidean spacetime can be written as [1]

$$(J^{-1}J_z)_{\bar{z}}+(J^{-1}J_y)_{\bar{y}}=0 \tag{1}$$

with $J\in \mathrm{SU}(N)$, $z=\frac{1}{2}(x_1+\mathrm{i}x_2)$ and $y=\frac{1}{2}(x_3+\mathrm{i}t)$ (the bar stands for complex conjugation).

Along with the elliptic system (1) we shall also consider its hyperbolic (or Minkowski) version, i.e. the system that is obtained by letting $y\to\frac{1}{2}(x_3+t)$ and $\bar{y}\to\frac{1}{2}(x_3-t)$. Note that in this case $\bar{y}$ is not the complex conjugate of y. There is also no relationship with the self-dual Yang-Mills equation in this case.

Letting $A=J^{-1}J_y$ and $B=J^{-1}J_z$ we obtain the equivalent system

$$B_{\bar{z}}+A_{\bar{y}}=0 \qquad B_y-A_z+[A,B]=0. \tag{2}$$

Although important solutions of system (2) have already been found, the study of these equations as a boundary value problem has not yet been undertaken. In this paper we shall try to consider this issue.

A natural problem for the hyperbolic case is the Cauchy problem: given the initial data $A(t=0)$ and $B(t=0)$, determine A and B for all times $t>0$ on the class of functions vanishing at the spatial infinity.

In the elliptic case these data by themselves do not define a well-posed problem. It turns out to be necessary to supplement them with a boundary condition: $A(x_1,x_2,x_3,t)$ and $B(x_1,x_2,x_3,t)$ tend to zero as t approaches infinity. This paper will be dedicated to the study of the above problems. We will use the inverse scattering method (IST). This is possible since (2) can be represented as the compatibility condition for [2]

$$L\mu\equiv[(k^2-1)/k\partial_1+\mathrm{i}(k^2+1)/k\partial_2+2\partial_3+U]\mu(z,\bar{z},y,\bar{y},k)=0 \tag{3a}$$

$$M\mu\equiv(-k\partial_{\bar{y}}+\partial_z+B)\mu(z,\bar{z},y,\bar{y},k)=0 \tag{3b}$$

where $U=A-B/k$, $\partial_i=\partial_{x_i}$ and $k=k_{\mathrm{R}}+\mathrm{i}k_{\mathrm{I}}$.

† On leave from University of Salamanca, Departamento de Matematicas, Salamanca 37008, Spain.

Indeed, the requirement that $LM-ML=0$ at once yields (2). Furthermore, from (3) it follows that

$$A=-(\mu_y\mu^{-1})(k=0) \qquad B=-(\mu_z\mu^{-1})(k=0). \tag{4}$$

2. Eigenfunctions of the operator L

In this section we shall consider the properties of the eigenfunctions of the operator L, in accordance with the IST method. We note that the operator L is common for both the elliptic and hyperbolic cases; hence, we do not need to distinguish between them at this stage. Differences will only appear when we consider the temporal evolution.

Let the function $\mu(x,k)$ solve $L\mu=0$ and $\mu(x,k)\to I$ as $k\to\infty$. It satisfies the integral equation

$$\mu(x,k)=I+\int G(x-x',k)U\mu(x',k)\,\mathrm{d}x' \tag{5}$$

where x stands for (x_1,x_2,x_3) and the Green function G is given by

$$G(x,k)=\frac{(k_R^2+k_I^2-1)\,\delta(f(x,k))}{4\pi(kz+\bar{z}/k)} \tag{6}$$

where δ stands for the usual delta function and $f(x,k)=2k_Ix_2-2k_Rx_1+(k_R^2+k_I^2-1)x_3$.

Comments:

(i) Equation (5) follows on noting that G solves $(L-U)G(x,k)=\delta(x)$ (G is fixed, demanding boundedness for all values of k) and taking Fourier transforms on both sides of (3a).

(ii) In this section we are working at $t=0$. Thus, (5) involves only initial data.

(iii) We assume that the integral equation (5) has solutions. This certainly requires some decay in the potentials; however, we have not considered the precise conditions for this to happen.

(iv) Equation (5) shows that the function $\mu(k)$ is in general non-holomorphic. Evaluating its departure from holomorphicity, which is given by $\partial\mu(k)/\partial\bar{k}$, is a central task in the inverse problem (see [3] for a good review on multidimensional IST).

We not identity $\partial\mu/\partial\bar{k}$. We find it convenient to rewrite (5) as

$$\mu(k)=I+G*U\mu$$

with the $*$ standing for convolution. It follows that

$$\partial\mu/\partial\bar{k}=(\partial G/\partial\bar{k})*U\mu+G*U\,\partial\mu/\partial\bar{k} \tag{7}$$

(there is a further contribution stemming from the term $\partial U/\partial\bar{k}$ that involves $\delta(k=0)$; however, this can be proved to be zero).

In order to proceed further we need to relate μ and $\partial\mu/\partial\bar{k}$. To this end we find it useful to introduce a function $N(x,k,l)$ that solves ($l\in\mathbb{R}$):

$$N(x,k,l)=\exp(\mathrm{i}lf(x,k))+\int G(x-x',k,l)U\cdot N(x',k,l)\,\mathrm{d}x'. \tag{8}$$

This eigenfunction is motivated by noticing that

$$\partial\mu/\partial\bar{k}=\int_{-\infty}^{\infty}N(x,k,l)F(k,l)\,\mathrm{d}l \tag{9}$$

where the scattering data $F(k, l)$ are defined by

$$F(k, l) = (\mathrm{i}/2\pi^2) lk \int \exp(-\mathrm{i}lf(x, k)) U \cdot \mu(x, k)\, \mathrm{d}x. \tag{10}$$

Now, multiply (8) by $\exp(-\mathrm{i}lf(x, k))$ and use $\delta(f(x)) \exp(\mathrm{i}lf(x)) = \delta(f(x))$ to obtain the crucial relationship

$$N(x, k, l) = \exp(\mathrm{i}lf(x, k)) \mu(x, k). \tag{11}$$

Hence we have

$$\partial\mu/\partial\bar{k} = \mu(k) S(f(x, k), k) \tag{12}$$

with

$$S(f(x, k)) = \int \exp(\mathrm{i}lf(x, k)) F(k, l)\, \mathrm{d}l. \tag{13}$$

In the foregoing analysis we assume that (5) has no homogeneous solutions. We now study how the above picture is modified in the presence of homogeneous modes. Let R_j be a such mode at the point $k = k_j$ $(j = 1 \ldots n)$, i.e. R_j solves as follows:

$$R_j(x) = \int G(x - x', k_j) U(x', k_j) \cdot R_j(x')\, \mathrm{d}x'. \tag{14}$$

In accordance with the Fredholm theory we assume that these points μ have poles with residues R_j. Hence, using $(\partial/\partial\bar{k})$ $(1/(k - k_j)) = \pi\delta(k - k_j)$ we find that the $\bar{\partial}$ derivative picks an extra term:

$$\partial\mu/\partial\bar{k} = \mu(k) S(f) + \sum R_j\, \delta(k - k_j). \tag{15}$$

Furthermore, a direct calculation proves that as long as R_j solves (14) then $R_j h(f(x, k_j))$ also solves (14) for any arbitrary h and thus it is also a homogeneous mode (provided $R_j h$ tends to zero at the spatial infinity). Thus we come across a very unusual phenomenon: the eigenvalues are degenerate with an infinite multiplicity. From a physical point of view this arbitrariness corresponds [4] to gauge freedom. This allows one to set $h = 1$ without loss of generality.

We close this section by giving the asymptotic form of μ at spatial infinity,

$$\mu \to I + (k_1^2 + k_2^2 - 1)/(k(kz + \bar{z}/k)) \int \exp(\mathrm{i}lf(x, k)) F(k, l)/\mathrm{i}l\, \mathrm{d}l \tag{16}$$

which follows by using partial integration in (5). This asymptotic expression shows that $J^{-1} \equiv \mu(k = 0)$ and J tend to I as $r = (x_1^2 + x_2^2 + x_3^2) \to \infty$, i.e. that A and B tend to zero at infinity. Note further that there is an extra decay in (16), since the integral is zero for any sufficiently regular F (this follows from the Riemann–Lebuesgue lemma). Thus, $\mu \to I + \mathrm{o}(1/r)$ as $r \to \infty$.

3. Temporal evolution of the scattering data

We now consider the temporal evolution of the scattering data, which is determined from $M\mu = 0$. Since the operator M involves ∂_t it follows that we have to separately consider the hyperbolic and elliptic cases. To be specific, consider the above equation along with the equation that follows upon operating on it with $\partial/\partial\bar{k}$, i.e. $M(\partial\mu/\partial\bar{k})=0$, and use our result (12) to obtain

$$\varepsilon F_t = \mathrm{i}l(k_\mathrm{R}^2+k_\mathrm{I}^2+1)F \tag{17}$$

with $\varepsilon=-1$ on the hyperbolic case and $\varepsilon=-\mathrm{i}$ in the elliptic case. Thus the evolution of the scattering data is given by

$$F(k,l,t)=F(k,l,t=0)\exp[(\mathrm{i}l/\varepsilon)(k_\mathrm{R}^2+k_\mathrm{I}^2+1)t]. \tag{18}$$

The boundary condition of the elliptic case requires $F_{\text{elliptic}}(k,l,0)=0$ for $l<0$.

On taking into account these expressions, we find that matrix S takes the form $S(\bar{k}(ky-\bar{z})-kz-\bar{y}),k)$ in terms of the original coordinates.

Comments:

(i) Equation (12) is sometimes derived in the literature [5] by noting that L, M are first-order operators, holomorphic with respect to k, and thus if μ solves $L\mu=0$, $M\mu=0$ then $\partial\mu/\partial\bar{k}$ solves the same equations. This yields at once a relationship such as (12) for some S. It has also been obtained by geometrical methods [6]. However, this kind of reasoning does not give any information on the specific form of S in terms of initial functions (our (10) and (13)). Neither do they regarding the temporal evolution of initial data. Hence, as far as we know most of the work considered here is new. It is now possible to solve the aforementioned Cauchy and boundary problems.

(ii) The solution of the direct and inverse problem uses a pure $\bar{\partial}$ problem, in the sense that it does not degenerate into a Riemann problem as happens for some (2+1)-dimensional reductions of the model (see [7]).

4. Constraints on the scattering data

We note that (in the elliptic case) there are some natural constraints on the scattering data. First, unimodularity of J yields that matrices A and B are traceless. Furthermore, Hermiticity of J implies that as long as μ solves (3) then $J^{-1}\mu^{-1}(-1/\bar{k})^\dagger$ solves (3). On using (4) along with the normalization of μ it follows that this function tends to I as k approaches infinity and hence that (on assuming uniqueness)

$$\mu(k)=J^{-1}\mu^{-1}(-1/\bar{k}). \tag{19}$$

We find upon insertion of this relationship into (12) that

$$\begin{aligned} &F(k,l)=(-k/\bar{k})F^\dagger(-1/\bar{k},k\bar{k}l)\\ &S(f(x,k),k)=(-1/\bar{k}^2)S^\dagger(f(x,-1/\bar{k}),-1/\bar{k}). \end{aligned} \tag{20}$$

We now collect the requirements on admissible initial data. We assume that

$$\begin{aligned} &A(t=0),\ B(t=0)\in L^1\cap L^2\\ &F(k,l,0)\in L^1\cap L^2\\ &F(0,l,0)=F(k,0,0)=0 \end{aligned} \tag{21}$$

(see (12) in connection with the last identity). The aforementioned properties yield at once

$$\begin{aligned} &S(f(x,0),0)=0\\ &S\to 0 \qquad \text{along the line } f(x,k)\to\infty. \end{aligned} \tag{22}$$

All this is common for both the elliptic and hyperbolic cases. In the elliptic case $F(k, l)$ is zero for $l<0$ and it satisfies condition (20). Hence, admissible initial data are far more restrictive in the elliptic case than in the hyperbolic one.

5. Solution of the inverse problem

It remains to give an expression for the solution of the inverse problem. This is accomplished by using the Cauchy formula for non-holomorphic functions,

$$\mu(k)=I+(1/2\pi\mathrm{i})\int\frac{(\partial\mu/\partial\bar{z})(z)}{z-k}\,\mathrm{d}z\,\mathrm{d}\bar{z} \tag{23}$$

where the first term comes from the integral of μ along the contour at infinity and taking into account the normalization of μ at infinity and the integral in the second runs over the whole complex plane of the coordinate z.

Equation (23) is to be supplemented by (15).

The solution of the boundary problem for (1) is now reduced to solving a set of linear problems. We now recall how this is done in the framework of the IST method. Given the Cauchy initial data $A(t=0)$, $B(t=0)$ (or $J(t=0)$, $\partial J/\partial t)(t=0))$ we first obtain $\mu(t=0)$ from (5). Then from (10) and (13) we have $F(t=0)$ and $S(t=0)$ (note that we are dropping the spatial dependence when unnecessary). The time evolution of F and S follows from (18). Next, (23) allows one to find $\mu(t)$ and finally we have $J(t)=\mu^{-1}(t, k=0)$. The entire process only involves solving linear problems.

Thus we have solved the boundary problems for (1) up to the task of solving the linear problems given by either ($3a$) or (5) and (23) (equivalently (15)), and this is indeed the sense in which the IST solves a boundary problem for a nonlinear equation.

Although solving (23) seems in general a non-trivial problem, we will now see that in the case of a pure triangular matrix S it can be solved with full generality [6]. For the sake of simplicity assume that we are working with 2×2 matrices and let

$$\mu=\begin{pmatrix}\mu_1 & \mu_3\\ \mu_2 & \mu_4\end{pmatrix} \qquad S=\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}. \tag{24}$$

Assume further that μ has no poles. Then (15) reduces to

$$\partial\mu_1/\partial\bar{k}=a\mu_1 \tag{25a}$$

$$\partial\mu_3/\partial\bar{k}=b\mu_1+c\mu_3 \tag{25b}$$

and similarly for μ_2 and μ_4. These equations are then supplemented by the normalization $\mu\to I$ as $k\to\infty$. Their solution is a trivial task. Indeed, the first equation has the general solution

$$\mu_1(k)=f(k)\exp\left((1/2\pi\mathrm{i})\int\frac{a(z)\,\mathrm{d}z\,\mathrm{d}\bar{z}}{z-k}\right) \tag{26}$$

with $f(k)$ holomorphic. The normalization of μ along with Liouville's theorem yields. $f\equiv1$. Substituting the above result into ($25b$) we finally get

$$\mu_3(k)=\left[g(k)+(1/2\pi\mathrm{i})\int\frac{b(z)}{z-k}\,\mathrm{d}z\,\mathrm{d}\bar{z}\exp\left((1/2\pi i)\int\frac{a(z')-c(z')}{z'-z}\,\mathrm{d}z'\,\mathrm{d}\bar{z}'\right)\right]$$
$$\times\exp\left((1/2\pi\mathrm{i})\int\frac{c(z)}{z-k}\,\mathrm{d}z\,\mathrm{d}\bar{z}\right) \tag{27}$$

where the function $g(k)$ depends holomorphically upon k and tends to 0 as $k \to \infty$. It follows that $g = 0$. Note also that $z, z' \in \mathbb{C}$.

Similar results hold for μ_2 and μ_4.

The opposite case, i.e. $F = S = 0$ and μ meromorphic (holomorphic except for some poles at $k = k_j$) can also be solved explicitly. The solutions that follow are the well-known soliton and instanton solutions. Since this issue has been extensively considered in the literature we will not elaborate further (see for example [8]). In this case system (15) reduces to an algebraic linear system and its solution is a matter of linear algebra.

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References

[1] Yang C 1977 *Phys. Rev. Lett.* **38** 1377
[2] Belavin A and Zakharov V 1978 *Phys. Lett.* **73B** 53
[3] Ablowitz M J and Fokas A 1983 *Proc. CIFMO School and Workshop, Oaxtepec, Mexico, 1982* ed K B Wolf *Lecture Notes in Physics* vol 189 (Berlin: Springer) pp 3–24
Fokas A and Ablowitz M J 1983 *Proc. CIFMO School and Workshop, Oaxtepec, Mexico, 1982* ed K B Wolf *Lecture Notes in Physics* vol 189 (Berlin: Springer) pp 137–58
[4] Chakravarty S Private conversation
[5] Beals R and Coifman R 1986 *Physica* **18D** 242; 1989 *Inverse Problems* **5** 131
Ablowitz M J, Costa D G and Tenenblat K 1987 *Studies Appl. Math.* **77** 37
[6] Chakravarty S *PhD Thesis* Pittsburgh University (unpublished)
Chakavarty S and Newman E T 1987 *J. Math. Phys.* **28** 334
[7] Manakov S and Zakharov V 1981 *Lett. Math. Phys.* **5** 247
Villarroel J 1989 *Inverse Problems* **5** 1157; 1990 *Studies Appl. Math.* **83** 211
[8] Forgacs P, Horvath Z and Palla L 1983 *Nucl. Phys.* B **229** 77